Learning in Games Where Agents Sample

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Mixed Strategies in Game Theory

Mixed strategies are fundamental for Game Theory:

- Many games **don't have** Nash equilibria in **pure strategies**:
  - Matching pennies;
  - Rock, paper, scissors...

Questions:

1. Do players really **randomize**? **How**?
2. Why do players choose the strategy that makes other players **indifferent**?

How do we interpret **mixed-strategy equilibria**?

Sampling in Games With a Continuum of Agents

Perfect foresight of other players' strategies is **key** in the definition of **equilibria**.

With **uncountably infinitely** many agents:

- Assumption is **debatable**.

Relax it: Assume agents only observe the actions of a **finite sample** of other agents.

Questions:

1. Set of **equilibria**?
2. Are **equilibria** going to be very different depending on **sample size**?

Summary of Results

1. For games where **agents sample**, we define a **solution concept**:
   - **Sampling Bayesian Equilibrium**.
2. Proof the **existence of SBE** in two classes of games:
   - Coordination **Global Games**,
     - Regime switch model: **Unique** interior equilibrium.
   - **Static** Games.
3. Present **comparative statics** on SBE.
4. **Asymptotic** behaviour of SBE [sample $\rightarrow \infty$]:
   - Mixed-strategy equilibria **are limits** of pure-strategy equilibria of **Sampling Games**.
   - **Consistent** with Nash's 'mass-action' interpretation.

Literature

Purification Literature:
Harsanyi (1973, IJGT); Radner & Rosenthal (1982, MOR); Aumann et al. (1983, MOR); Govindan et al. (2003, GEB); Bhaskar et al. (2008, RED); Barelli & Duggan (2015, GEB)...

Contribution: Purification under **endogenous information**:

$$\text{Types/signals' distribution} \leftrightarrow \text{Agents' actions}$$

Global Games:
Carlsson & van Dame (1993, Econom.); Morris & Shin (1999); Morris & Shin (2006); Angeletos et al. (2006, JPE); Angeletos & Werning (2006; AER)...

Contribution: **Endogenizing** private signals.

Outline

Outline

Basic Setup

Regime Switch Model (Morris & Shin 2000):

Continuum of Agents and a **Regime**:

- Agents **indexed** by $i \in [0, 1]$.
- Binary **action set**:
  - **Attack** the regime ($a_i = 1$);
  - **Don't attack** the regime ($a_i = 0$).
- **Mass** of agents attacking: $A \in [0, 1]$.
- Regime's **strength**: $\theta \in \mathbb{R}$.
- **Cost** of attacking: $C \in (0, 1)$

Payoffs

Payoffs:

$a_i = 0$ [Agents doesn't attack]: **Normalized** to 0.

$a_i = 1$ [Agent attacks]:

$A > \theta$ [Regime is **overthrown**]: $1 - C$.
$A \leq \theta$ [Regime **survives**]: $-C$.

Thus,

$$U(a_i, A, \theta) = a_i(\mathbb{I}\{A > \theta\} - C)$$

Remark: Weak **monotonicity** of $U(\cdot)$ w.r.t. A

$\Rightarrow$ Incentive to **coordinate**.

Applications

1. **Bank Runs:**
   Withdrawals>Reserves ⇒ Bank **defaults** (regime switch)
   [Goldstein & Pauzner (2005); Cañon & Margaretic (2014)...]
2. **Currency Crises:**
   Sufficiently many speculators **attack the currency** ⇒ Central bank **abandons peg**.
   [Obstfeld (1996); Morris and Shin (1998)...]
3. **Investments with Complementarities:**
   Project's profitability depends on **number of investors**.
   [Dasgupta (2007)]

Outline

Incomplete Exogenous Information (Morris and Shin 2000)

Information Structure:

- Regime's **strength**: $\theta \in \mathbb{R}$. [Unknown]
  1. Nature draws θ from distribution P (agent's **common prior**);
  2. **Public Signal:** $y = \theta + \sigma_y \varepsilon$, where $\varepsilon \sim \mathcal{N}(0, 1)$;
  3. **Private Signal:** $x_i = \theta + \sigma_x \varepsilon_i$, where $\varepsilon_i \sim \mathcal{N}(0, 1)$;
- Cost of attacking: $C \in (0, 1)$ [Common Knowledge]
- **Perfect foresight** on strategy profile of the continuum.

Equilibria:

- Uniqueness $\Leftrightarrow \sigma_x < \sigma_z^2 \sqrt{2\pi}$

Incomplete Endogenous Information

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  1. Nature draws θ from distribution P (agent's **common prior**);
  2. **Public Signal:** $y = \theta + \sigma_y \varepsilon$, where $\varepsilon \sim \mathcal{N}(0, 1)$;
- Cost of attacking: $C \in (0, 1)$ [Common Knowledge]
- Mass of agents attacking: $A \in [0, 1]$. [Agent only observes action profile of finite i.i.d sample of players]
  3. **Private Signal:** X_i: # of attackers in player i's sample.

Introducing Sampling in the Game

Timing for **individual** agent:

1. **Public signal** Y and **cost** C are observed.
   - **Posterior** on regime's strength is formed: $g_{\theta|Y}$
2. **Prior** on A (size of attack): $p_0(A|Y, C)$.
   a) **Non-degenerate**;
   b) Has a **probability density**.
3. Agent **observes** X_i from her **sample** (**size** n).
4. **Bayesian updating** of beliefs: $p_i(A|Y, C, X_i)$.
5. Agent decides to **attack** ($a_i = 1$) or **not** ($a_i = 0$).

Threshold Strategies

Agent **attacks** if and only if:

$$\mathbb{E}[U(1, A, \theta)|Y, C, X_i] = \Pr[A > \theta|Y, C, X_i] - C > 0$$

Therefore:

Agents follow a **threshold strategy** of the form:

Proposition 1: $a_i = 1 \Leftrightarrow X_i > \bar{X}_n(Y, C)$

Outline

Sampling Bayesian Equilibrium

Circularity between **action profiles** and **beliefs**:

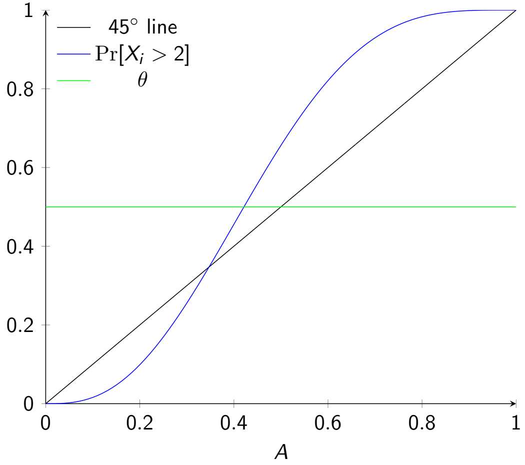
1. Aggregate attack (A) **induces a distribution** of private signals: $X_i \sim B(n, A)$.
2. Distribution of private signals, according to $\bar{X}_n(Y, C)$, **determines** A.

Coherency:
Aggregate attack **equals** the probability that X_i realizes above threshold $\bar{X}_n(Y, C)$:

$$A = \Pr[X_i > \bar{X}_n(Y, C)] \tag{2.1}$$

Example: Solutions to Equation 2.1

Given Y, C and $p_0(Y, C)$, suppose $\bar{X}_6(Y, C) = 2$.



Solutions to Equation 2.1:

Proposition 2 [Unique Interior Solution]:

For sufficiently large n,

$$\exists! A_n^* \in (0,1) \text{ that satisfies } \textbf{coherency}.$$

Remark:

- **Multiplicity of Equilibria:** $A = 0$ and $A = 1$ also satisfy **coherency**

What happens to A_n^* as:

1. **Public signal** Y changes?
2. **Cost** C changes?

Comparative Statics w.r.t Y

Imposing some **monotonicity conditions** on the prior:

Conditions

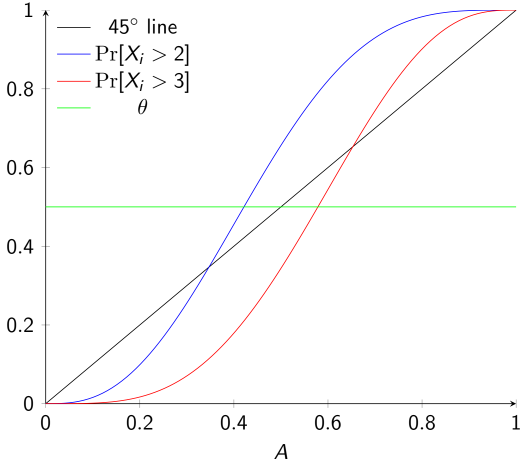
1. Subjective **probability of attack succeeding** decreases:
$$Y' > Y \Rightarrow \Pr[A > \theta | Y, C, X_i] > \Pr[A > \theta | Y', C, X_i]$$

2. **Cutoff** of threshold strategy is weakly increasing in Y:
$$\bar{X}_n(Y', C) \geq \bar{X}_n(Y, C)$$

What happens to the **SBE**?

Example: Comparative Statics

Given $Y' > Y$, suppose $\bar{X}_6(Y, C) = 2$, $\bar{X}_6(Y', C) = 3$.



Outline

Mixed Strategies under Complete Information

Complete Information:
Suppose agents had **perfect foresight** on action profiles.

Mixed Strategies:

- Players must be **indifferent** between attacking or not to **randomize**.
- Size of the attack A^* must satisfy:

$$\Pr[A^* > \theta | Y, C] - C = 0 \Rightarrow \int_0^{A^*} g_{\theta|Y} \, \mathrm{d}\theta = C$$

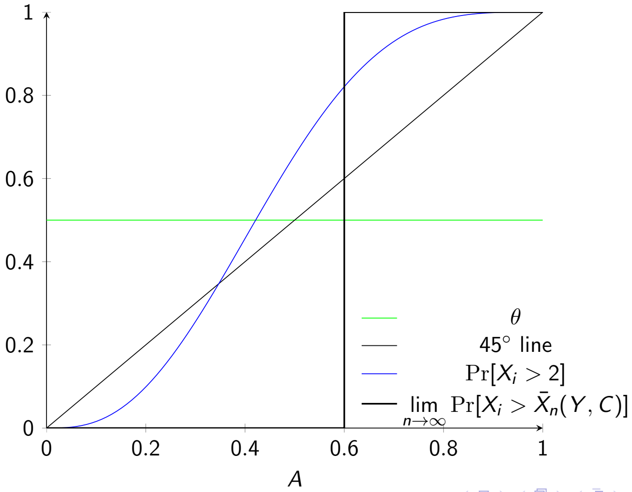
Purification Result

Theorem 1: If prior $p_0(A|Y, C)$ is **continuous and strictly positive** in $[0, 1]$, then:

$$\lim_{n \to \infty} A_n^* = A^*$$

Example: Asymptotic Behaviour of SBE

Given Y, C and $p_0(Y, C)$, suppose $\bar{X}_6(Y, C) = 2$ and $A^* = 0.6$.



What Drives the Purification Result?

What happens to the **posteriors** as $n \to \infty$?

Bernstein-von Mises Theorem [B-vM]:

1. Asymptotically **independent of the prior**.
2. Asymptotically **normal**:
   - Mean: **Maximum Likelihood Estimator**.
   - Variance: **Cramér-Rao's** lower bound.

Bernstein-von Mises Theorem

B-vM Theorem in our Setting

In the model:

1. MLE is the **sample average**: $\hat{A}_{MLE} = n^{-1}X_i$.
2. **Cramér-Rao's** lower bound: $n^{-2}A(1-A)$

Bernstein-von Mises' Theorem implies:

$$\left|\left|p_i(A|Y, C, X_i) - \mathcal{N}\left(n^{-1}X_i,\ n^{-2}A(1-A)\right)\right|\right|_{TV} \xrightarrow{P} 0$$

Complete Information is Restored as $n \to \infty$

Asymptotically, **posterior beliefs** have the form:

$$\mathcal{N}\left(n^{-1}X_i,\ n^{-2}A(1-A)\right)$$

Thus, as $n \to \infty$:

1. **[Law of Large Numbers]**: $n^{-1}X_i \xrightarrow{P} A$

   "Means of the posteriors **cluster** around A."

2. $n^{-2}A(1-A) \to 0$

   "**Tails** of the posteriors **become thinner** and thinner."

"More and more agents are more and more sure of the **true size** of the attack A"

Outline

Static Game: Players and Payoffs

Notation:

- N: finite number of players;
- A_i: finite set of **pure actions** for player i;
- $\pi_i : \times_{i=1}^{N} A_i \to \mathbb{R}$: **payoff function** of player i;
- $\Gamma = \left(A_i, \pi_i\right)_{i=1}^{N}$: finite **static game** in strategic form.

Question:

Can we **approximate** the **mixed-strategy** equilibria of Γ with **SBE** of perturbed games?

Outline

Sampling Game Γ_n

Players and Payoffs:

- $\{1, 2, \cdots N\}$: **classes** of players;
- Each class is composed by **unit-mass continuum** of identical agents:
  - Same **action set**: A_i.
  - Same **payoff function**: $\pi_i : \times_{i=1}^{N} A_i \to \mathbb{R}$

Notation:

- $\eta^i(a_i)$: **Share of players** in class i playing strategy a_i.
- $\eta^i = \left(\eta^i(a_1), \eta^i(a_2), \cdots \eta^i(a_{m_i})\right)$: **Shares** for class i
- $\eta^{-i} = \left(\eta^1, \cdots \eta^{i-1}, \eta^{i+1} \cdots \eta^N\right)$

Information Structure

Information Structure:

- Players and Payoffs: Γ [Common Knowledge]
- Shares: η^{-i}
  [Agent only observes action profile of i.i.d samples of the $(N-1)$ remaining classes]
  - **Private Signal:**
    - $X_j^i(a_j)$: # of class j players playing strategy a_j seen by a player of class i in her sample.
    - $X_j^i = (X_j^i(a_1), X_j^i(a_2), \cdots X_j^i(a_{m_j}))$
    - $X^i = (X_1^i, \cdots X_{i-1}^i, X_{i+1}^i \cdots X_N^i)$
  - n: sample size for all samples.

Sampling in the Game

Timing for **individual** agent:

1. Player **classes and payoffs** in Γ are observed.
2. **Prior** on η^{-i} (shares): $p_i^0(\eta^{-i} | \Gamma)$.
   a) **Non-degenerate**;
   b) Has a **probability density**.
3. Agent **observes** X^i from her **samples**.
4. **Bayesian updating** of beliefs: $p_i(\eta^{-i} | X^i, \Gamma)$.
5. Agent is **grouped randomly** with players of the other $N-1$ classes and chooses an action $a_i \in A_i$.

Expected Payoffs and Optimal Actions

Expected Payoff:
Expected payoff of action a_i, **conditional** on X^i:

$$U_i(a_i|X^i) = \int \Big(\sum_{a_{-i} \in A_{-i}} \eta^{-i}(a_{-i}) \pi_i(a_i, a_{-i}) \Big) p_i(\eta^{-i}|X^i, \Gamma)\, \mathrm{d}\eta^{-i}$$

Optimal Actions:
The set of optimal actions given signal X_i:

$$\mathrm{OA}(X^i) = \arg\max_{a_i \in A_i} U_i(a_i|X^i)$$

SBE of Γ_n in Symmetric Strategies

Let:

- $\mathbb{X}^i$: Support of random vector X^i.
- $s \equiv \{s_i : \mathbb{X}^i \to A_i\}_{i=1}^N$ be a **symmetric** strategy profile.
- $\eta \equiv \left(\eta^i\right)_{i=1}^N$ be a collection of **share vectors**.

Definition:

(s, η) is a **SBE** in **symmetric** strategies if for all i, $a_i \in A_i$, and $x_i \in \mathbb{X}^i$:

1. $s_i(x^i) \in \mathrm{OA}(x^i)$;
2. $\eta^i(a_i) = \Pr\left[X^i \in s_i^{-1}(a_i)\right]$
   - Where $X^i \sim \otimes_{j \neq i} \mathrm{Multi}\left(n; \eta^j(a_1), \eta^j(a_2) \cdots \eta^j(a_{m_j})\right)$

Existence Result

Theorem 2: For every static game Γ and every n,

Γ_n has a **SBE** in **symmetric strategies**.

Outline

Purification in 2×2 games

Suppose Γ is of the form:

		Player B	
		1	2
Player A	1	(a_{11}, b_{11})	(a_{12}, b_{12})
	2	(a_{21}, b_{21})	(a_{22}, b_{22})

Purification in 2×2 games

Let σ be an **equilibrium** of Γ in which no player uses a **weakly dominated strategy**.

Theorem 3: If $p_0(\eta^{-i} | \Gamma)$ is **strictly positive and continuous** in Δ_1,

$\Rightarrow$ exists a **sequence of SBE** in symmetric strategies $(s_n, \eta_n)_{n=1}^{\infty}$ of **sampling games** $(\Gamma_n)_{n=1}^{\infty}$, such that:

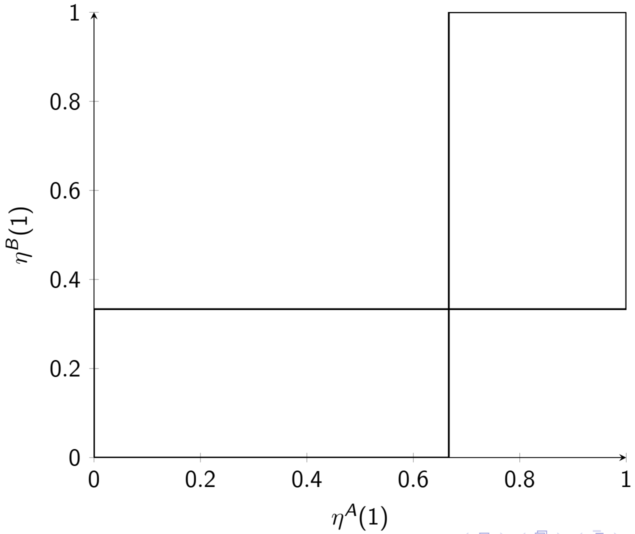
$$\lim_{n \to \infty} \eta_n = \sigma$$

Example: Battle of the Sexes

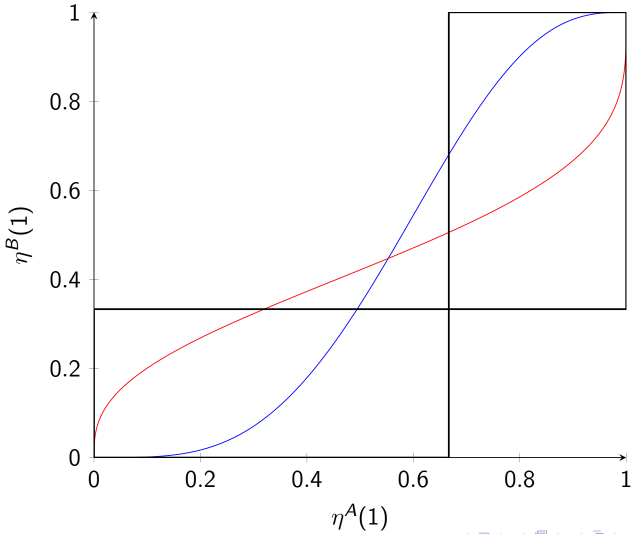
Suppose Γ is of the form:

		Player B	
		1	2
Player A	1	$(2, 1)$	$(0, 0)$
	2	$(0, 0)$	$(1, 2)$

Example: Battle of the Sexes



Example: Battle of the Sexes



Conclusion

1. For games where **agents sample**, we defined a **solution concept**:
   - **Sampling Bayesian Equilibrium**.
2. Showed the **existence of SBE** in two classes of games:
   - Coordination **Global Games**,
     - Regime switch model: **Unique** interior equilibrium.
   - **Static** Games.
3. Presented **comparative statics** on SBE.
4. **Asymptotic** behaviour of SBE [sample $\to \infty$]:
   - Mixed-strategy equilibria **are limits** of pure-strategy equilibria of **Sampling Games**.
   - **Consistent** with Nash's 'mass-action' interpretation.

▸ Nash

Thank you!

Conditions on the Prior

Assumption:

$$Y' > Y \Rightarrow p_0(A|Y, C) \succsim_{\text{MLR}} p_0(A|Y', C)$$

MLR **implies** FSD:

$$p_0(A|Y, C) \succsim_{\text{MLR}} p_0(A|Y', C) \Rightarrow p_0(A|Y, C) \succsim_{\text{FSD}} p_0(A|Y', C)$$

MLR is **preserved** under **Bayesian updating** (Klemens; 2007)

Comparative Statics w.r.t Y

Bernstein-von Mises Theorem

Let:

- $\{X_1, X_2 \cdots X_n\}$: **i.i.d. observations** from density $f(\theta_0)$.
- $\hat{\theta}_n$: **ML estimator**.
- Π: The prior measure, with density π which is **continuous and positive** in a neighborhood of θ_0.

Then, if $\Pi(\cdot|X_1, X_2 \cdots X_n)$ is the **posterior distribution**:

$$\left|\left|\Pi(\cdot|X_1, X_2 \cdots X_n) - \mathcal{N}\left(\hat{\theta}_n, n^{-1} i^{-1}(\theta_0)\right)\right|\right|_{TV} \xrightarrow{P_{\theta_0}} 0$$

Where,

- $i(\theta_0)$: Fisher's Information
- $\|\cdot\|_{TV}$: Total Variation Distance

Asymptotic Behaviour of Beliefs

Nash's Rationale

"**Mass-action** interpretation":

- "[A]ssume that there is a **population** (in the sense of **statistics**) of participants[...]"
- "[P]articipants [...] **accumulate empirical information** on the relative advantages of [...] strategies [...]"
- "**[A]verage behaviour** in each of the populations form an **equilibrium point**[...].
- "[W]e can only expect some sort of **approximate equilibrium** since the information, [...] will be imperfect."

[Nash (1950); pages 21–23]